

# DECOMPOSITION OF POINTWISE FINITE-DIMENSIONAL PERSISTENCE MODULES

WILLIAM CRAWLEY-BOEVEY

**ABSTRACT.** We show that a persistence module (for a totally ordered indexing set) consisting of finite-dimensional vector spaces is a direct sum of interval modules. The result extends to persistence modules with the descending chain condition on images and kernels.

## 1. INTRODUCTION

We fix a base field  $k$  and a totally ordered indexing set  $(R, <)$ , and assume throughout that  $R$  has a countable dense subset. A *persistence module*  $V$  is a functor from  $R$ , considered in the natural way as a category, to the category of vector spaces. Thus it consists of vector spaces  $V_t$  for  $t \in R$  and linear maps  $\rho_{ts} : V_s \rightarrow V_t$  for  $s \leq t$ , satisfying  $\rho_{ts}\rho_{sr} = \rho_{tr}$  for all  $r \leq s \leq t$  and  $\rho_{tt} = 1_{V_t}$  for all  $t$ .

Persistence modules indexed by the natural numbers were introduced in [8] as algebraic objects underlying persistent homology. But other indexing sets are of interest, especially the real numbers—see [1, 2, 3, 6].

A subset  $I \subseteq R$  is an *interval* if it is non-empty and  $r \leq s \leq t$  with  $r, t \in I$  implies  $s \in I$ . The corresponding *interval module*  $V = k_I$  is given by  $V_t = k$  for  $t \in I$ ,  $V_t = 0$  for  $t \notin I$  and  $\rho_{ts} = 1$  for  $s, t \in I$  with  $s \leq t$ . We say that a persistence module  $V$  is *pointwise finite-dimensional* if all  $V_t$  are finite-dimensional. Our aim is to give a short proof of the following result.

**Theorem 1.1.** *Any pointwise finite-dimensional persistence module is a direct sum of interval modules.*

More generally, we say that  $V$  has the *descending chain condition on images and kernels* provided that for all  $t, s_1, s_2, \dots \in R$  with  $t \geq s_1 > s_2 > \dots$ , the chain

$$V_t \supseteq \operatorname{Im} \rho_{ts_1} \supseteq \operatorname{Im} \rho_{ts_2} \supseteq \dots$$

stabilizes, and for all  $t, r_1, r_2, \dots \in R$  with  $t \leq \dots < r_2 < r_1$ , the chain

$$V_t \supseteq \operatorname{Ker} \rho_{r_1 t} \supseteq \operatorname{Ker} \rho_{r_2 t} \supseteq \dots$$

stabilizes.

---

Mathematics Subject Classification (2010): 16G20.

**Theorem 1.2.** *Any persistence module with the descending chain condition on images and kernels is a direct sum of interval modules.*

Our proof uses a variant of the ‘functorial filtration’ method, which we learnt from [7]. The situation here is rather simpler, however, with no ‘band modules’, and we haven’t found it necessary to exploit functoriality.

Since the endomorphism ring of an interval module  $k_I$  is  $k$ , a local ring, it follows from the Krull-Remak-Schmidt-Azumaya theorem that the multiplicity of  $k_I$  as a summand of  $V$  is uniquely determined. Our method of proof shows that this is also the dimension of a certain vector space  $V_I^+/V_I^-$ .

One can study purity for persistence modules, as they form a locally finitely presented abelian category, see for example [4, §1.2, Theorem]. In this language, the descending chain condition on images and kernels corresponds to  $\Sigma$ -pure-injectivity—see [4, §3.5, Theorem 1] for four equivalent conditions. (For example, the spaces  $\text{Im } \rho_{ts}$  and  $\text{Ker } \rho_{rt}$  for  $s \leq t \leq r$  are subgroups of finite definition of  $V_t = \text{Hom}(\text{Hom}(-, t), V)$ , so condition (3) of the cited theorem implies the descending chain condition on images and kernels; conversely, this chain condition passes to products of copies of  $V$ , and then Theorem 1.2 implies condition (4) of the cited theorem.)

I would like to thank Vin de Silva for introducing me to persistence modules, Michael Lesnick for prompting me to write this paper and invaluable comments on a first draft, and both of them for many stimulating questions.

## 2. CUTS

Let  $V$  be a persistence module with the descending chain condition on images and kernels, and let  $t \in R$ . A *cut* for  $R$  is a pair  $c = (c^-, c^+)$  of subsets of  $R$  such that  $R = c^- \cup c^+$  and  $s < t$  for all  $s \in c^-$  and  $t \in c^+$ . If  $c$  is a cut with  $t \in c^+$ , we define subspaces  $\text{Im}_{ct}^- \subseteq \text{Im}_{ct}^+ \subseteq V_t$  by

$$\text{Im}_{ct}^- = \bigcup_{s \in c^-} \text{Im } \rho_{ts}, \quad \text{Im}_{ct}^+ = \bigcap_{\substack{s \in c^+ \\ s \leq t}} \text{Im } \rho_{ts},$$

and if  $c$  is a cut with  $t \in c^-$ , we define subspaces  $\text{Ker}_{ct}^- \subseteq \text{Ker}_{ct}^+ \subseteq V_t$  by

$$\text{Ker}_{ct}^- = \bigcup_{\substack{r \in c^- \\ t \leq r}} \text{Ker } \rho_{rt}, \quad \text{Ker}_{ct}^+ = \bigcap_{r \in c^+} \text{Ker } \rho_{rt}.$$

By convention  $\text{Im}_{ct}^- = 0$  if  $c^-$  is empty and  $\text{Ker}_{ct}^+ = V_t$  if  $c^+$  is empty.

**Lemma 2.1.** *Let  $c$  be a cut.*

- (a) *If  $t \in c^+$ , then  $\text{Im}_{ct}^+ = \text{Im } \rho_{ts}$  for some  $s \in c^+$  with  $s \leq t$ .*
- (b) *If  $t \in c^-$  and  $c^+ \neq \emptyset$ , then  $\text{Ker}_{ct}^+ = \text{Ker } \rho_{rt}$  for some  $r \in c^+$ .*

*Proof.* (a) Suppose that  $\text{Im}_{ct}^+ \neq \text{Im} \rho_{ts}$  for all  $s \in c^+$  with  $s \leq t$ . Let  $s_1 = t$ . Since  $\text{Im}_{ct}^+ \neq \text{Im} \rho_{ts_1}$ , there is  $s_2 \in c^+$  with  $\text{Im} \rho_{ts_2}$  strictly contained in  $\text{Im} \rho_{ts_1}$ . Similarly, since  $\text{Im}_{ct}^+ \neq \text{Im} \rho_{ts_2}$ , there is  $s_3 \in c^+$  with  $\text{Im} \rho_{ts_3}$  strictly contained in  $\text{Im} \rho_{ts_2}$ , and so on. Then the chain  $\text{Im} \rho_{ts_1} \supset \text{Im} \rho_{ts_2} \supset \dots$  doesn't stabilize, which is a contradiction. Part (b) is similar.  $\square$

**Lemma 2.2.** *Let  $c$  be a cut and  $s \leq t$ .*

- (a) *If  $s, t \in c^+$ , then  $\rho_{ts}(\text{Im}_{cs}^\pm) = \text{Im}_{ct}^\pm$ , and*
- (b) *If  $s, t \in c^-$ , then  $\rho_{ts}^{-1}(\text{Ker}_{ct}^\pm) = \text{Ker}_{cs}^\pm$ , so  $\rho_{ts}(\text{Ker}_{cs}^\pm) \subseteq \text{Ker}_{ct}^\pm$ .*

*Proof.* It is clear that  $\rho_{ts}(\text{Im}_{cs}^+) \subseteq \text{Im}_{ct}^+$ . Now  $\text{Im}_{cs}^+ = \text{Im} \rho_{sr}$  for some  $r \in c^+$  with  $r \leq s$ . Then

$$\rho_{ts}(\text{Im}_{cs}^+) = \rho_{ts}(\text{Im} \rho_{sr}) = \text{Im} \rho_{tr} \supseteq \text{Im}_{ct}^+.$$

The rest is straightforward.  $\square$

### 3. INTERVALS

If  $I$  is an interval, there are uniquely determined cuts  $\ell$  and  $u$  with  $I = \ell^+ \cap u^-$ . Explicitly,

$$\begin{aligned} \ell^- &= \{t : t < s \text{ for all } s \in I\}, & \ell^+ &= \{t : t \geq s \text{ for some } s \in I\}, \\ u^+ &= \{t : t > s \text{ for all } s \in I\}, & u^- &= \{t : t \leq s \text{ for some } s \in I\}. \end{aligned}$$

For  $t \in I$  we define  $V_{It}^- \subseteq V_{It}^+ \subseteq V_t$  by

$$\begin{aligned} V_{It}^- &= (\text{Im}_{\ell t}^- \cap \text{Ker}_{ut}^+) + (\text{Im}_{\ell t}^+ \cap \text{Ker}_{ut}^-), \text{ and} \\ V_{It}^+ &= \text{Im}_{\ell t}^+ \cap \text{Ker}_{ut}^+. \end{aligned}$$

**Lemma 3.1.** *For  $s \leq t$  in  $I$  we have  $\rho_{ts}(V_{Is}^\pm) = V_{It}^\pm$ , and the map*

$$\bar{\rho}_{ts} : V_{Is}^+ / V_{Is}^- \rightarrow V_{It}^+ / V_{It}^-$$

*is an isomorphism.*

*Proof.* This follows from Lemma 2.2. For example, if  $h \in \text{Im}_{\ell t}^+ \cap \text{Ker}_{ut}^-$ , then  $h = \rho_{ts}(g)$  for some  $g \in \text{Im}_{\ell s}^+$ . But then

$$g \in \rho_{ts}^{-1}(h) \subseteq \rho_{ts}^{-1}(\text{Ker}_{ut}^-) = \text{Ker}_{us}^-$$

so  $g \in V_{It}^-$  and  $h = \rho_{ts}(g) \in \rho_{ts}(V_{It}^-)$ .

The map  $\bar{\rho}_{ts}$  is clearly surjective. To show it is injective, we show that  $V_{Is}^+ \cap \rho_{ts}^{-1}(V_{It}^-) \subseteq V_{Is}^-$ . Again this follows from Lemma 2.2. Suppose that  $g \in V_{Is}^+$  and that  $h = \rho_{ts}(g) \in V_{It}^-$ . Then  $h = h_1 + h_2$  with  $h_1 \in \text{Im}_{\ell t}^- \cap \text{Ker}_{ut}^+$  and  $h_2 \in \text{Im}_{\ell t}^+ \cap \text{Ker}_{ut}^-$ . Then  $h_1 = \rho_{ts}(g_1)$  for some  $g_1 \in \text{Im}_{\ell s}^- \cap \text{Ker}_{us}^+$ . Now  $\rho_{ts}(g - g_1) = h_2 \in \text{Ker}_{ut}^-$ , so  $g - g_1 \in \text{Ker}_{us}^-$ . Also  $g - g_1 \in \text{Im}_{\ell s}^+$ , so  $g \in V_{Is}^-$ .  $\square$

**Lemma 3.2.** *Any interval  $I$  contains a countable subset  $S$  which is coinital, meaning that for all  $t \in I$  there is  $s \in S$  with  $s \leq t$ .*

*Proof.* If  $I$  has a minimum element  $m$ , then  $\{m\}$  is coinital, so suppose otherwise. By our standing assumption,  $R$  has a countable subset  $X$  which is dense, that is, for all  $r < t$  in  $R$ , there is  $s \in X$  with  $r \leq s \leq t$ . Now  $I \cap X$  is coinital in  $I$ , for if  $t \in I$ , then since  $I$  has no minimum element there is  $r \in I$  with  $r < t$ . By density there is  $s \in X$  with  $r \leq s \leq t$ . But then  $s \in I \cap X$  and  $s \leq t$ .  $\square$

#### 4. INVERSE LIMITS

Let  $I$  be an interval. For  $s \leq t$  in  $I$ ,  $\rho_{ts}$  induces maps  $V_{Is}^\pm \rightarrow V_{It}^\pm$ , so one can consider the inverse limit

$$V_I^\pm = \varprojlim_{t \in I} V_{It}^\pm.$$

Note that to fit with the conventions of [5, Chap. 0, §13.1], one should use the opposite ordering on  $I$ . Letting  $\pi_t : V_I^+ \rightarrow V_{It}^+$  denote the natural map, one can identify

$$V_I^- = \bigcap_{t \in I} \pi_t^{-1}(V_{It}^-) \subseteq V_I^+.$$

**Lemma 4.1.** *For any  $t \in I$ , the induced map*

$$\bar{\pi}_t : V_I^+ / V_I^- \rightarrow V_{It}^+ / V_{It}^-$$

*is an isomorphism.*

*Proof.* If  $s \leq t$ , then  $\rho_{ts}(V_{Is}^-) = V_{It}^-$  by Lemma 3.1. It follows that the system of vector spaces  $V_{It}^-$  for  $t \in I$ , with transition maps  $\rho_{ts}$ , satisfies the Mittag-Leffler condition [5, Chap. 0, (13.1.2)]. Now, thanks to Lemma 3.2, the hypotheses of [5, Chap. 0, Prop. 13.2.2] hold for the system of exact sequences

$$0 \rightarrow V_{It}^- \rightarrow V_{It}^+ \rightarrow V_{It}^+ / V_{It}^- \rightarrow 0,$$

and hence the sequence

$$0 \rightarrow V_I^- \rightarrow V_I^+ \rightarrow \varprojlim_{t \in I} V_{It}^+ / V_{It}^- \rightarrow 0$$

is exact. By Lemma 3.1, the maps  $\bar{\rho}_{ts}$  are all isomorphisms, so the inverse limit on the right hand side is isomorphic to  $V_{It}^+ / V_{It}^-$  for all  $t \in I$ , giving the result.  $\square$

#### 5. SUBMODULES

For each interval  $I$ , choose a vector space complement  $W_I^0$  to  $V_I^-$  in  $V_I^+$ . For  $t \in I$ , the restriction of  $\pi_t$  to  $W_I^0$  is injective by Lemma 4.1.

**Lemma 5.1.** *The assignment*

$$W_{It} = \begin{cases} \pi_t(W_I^0) & (t \in I) \\ 0 & (t \notin I) \end{cases}$$

*defines a submodule  $W_I$  of the persistence module  $V$ .*

*Proof.* If  $s \leq t$  in  $I$ , then  $\rho_{ts}\pi_s = \pi_t$ , so  $\rho_{ts}(W_{Is}) = W_{It}$ . Also, if  $s \leq t$  with  $s \in I$  and  $t \notin I$  then  $t \in u^+$ , so  $W_{Is} \subseteq V_{Is}^+ \subseteq \text{Ker}_{us}^+ \subseteq \text{Ker } \rho_{ts}$ .  $\square$

**Lemma 5.2.**  $V_{It}^+ = W_{It} \oplus V_{It}^-$  for all  $t \in I$ .

*Proof.* This follows from Lemma 4.1.  $\square$

**Lemma 5.3.**  $W_I$  is isomorphic to a direct sum of copies of the interval module  $k_I$ .

*Proof.* Choose a basis  $B$  of  $W_I^0$ . For  $b \in B$ , the elements  $b_t = \pi_t(b)$  are non-zero and satisfy  $\rho_{ts}(b_s) = b_t$  for  $s \leq t$ , so they span a submodule  $S(b)$  of  $W_I$  which is isomorphic to  $k_I$ . Now  $\{b_t : b \in B\}$  is a basis of  $W_{It}$ , for all  $t$ , so  $W_I = \bigoplus_{b \in B} S(b)$ .  $\square$

## 6. SECTIONS

It remains to prove that  $V$  is the direct sum of the submodules  $W_I$  as  $I$  runs through all intervals. We prove this using what we call in this paper a ‘section’. We work first in an arbitrary vector space.

By a *section* of a vector space  $U$  we mean a pair  $(F^-, F^+)$  of subspaces  $F^- \subseteq F^+ \subseteq U$ . We say that a set  $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$  of sections of  $U$  is *disjoint* if for all  $\lambda \neq \mu$ , either  $F_\lambda^+ \subseteq F_\mu^-$  or  $F_\mu^+ \subseteq F_\lambda^-$ ; that it *covers*  $U$  provided that for all subspaces  $X \subseteq U$  with  $X \neq U$  there is some  $\lambda$  with

$$X + F_\lambda^- \neq X + F_\lambda^+;$$

and that it *strongly covers*  $U$  provided that for all subspaces  $Y, Z \subseteq U$  with  $Z \not\subseteq Y$  there is some  $\lambda$  with

$$Y + (F_\lambda^- \cap Z) \neq Y + (F_\lambda^+ \cap Z).$$

**Lemma 6.1.** Suppose that  $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$  is a set of sections which is disjoint and covers  $U$ . For each  $\lambda \in \Lambda$ , let  $W_\lambda$  be a subspace with  $F_\lambda^+ = W_\lambda \oplus F_\lambda^-$ . Then  $U = \bigoplus_{\lambda \in \Lambda} W_\lambda$ .

*Proof.* Given a relation  $c_{\lambda_1} + \dots + c_{\lambda_n} = 0$ , by disjointness we may assume that  $F_{\lambda_i}^+ \subseteq F_{\lambda_1}^-$  for all  $i > 1$ . But then  $c_{\lambda_1} = -\sum_{i>1} c_{\lambda_i} \in F_{\lambda_1}^-$ , so it is zero. Thus the sum is direct.

Now let  $X = \bigoplus_{\lambda \in \Lambda} W_\lambda$  and suppose that  $X \neq U$ . By the covering property, there is  $\lambda$  with  $X + F_\lambda^- \neq X + F_\lambda^+$ . But then  $X + F_\lambda^+ = X + W_\lambda + F_\lambda^- \subseteq X + F_\lambda^-$ , since  $W_\lambda \subseteq X$ , a contradiction.  $\square$

**Lemma 6.2.** If  $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$  is a set of sections which is disjoint and covers  $U$ , and  $\{(G_\sigma^-, G_\sigma^+) : \sigma \in \Sigma\}$  is a set of sections which is disjoint and strongly covers  $U$ , then the set

$$\{(F_\lambda^- + G_\sigma^- \cap F_\lambda^+, F_\lambda^- + G_\sigma^+ \cap F_\lambda^+) : (\lambda, \sigma) \in \Lambda \times \Sigma\}.$$

is disjoint and covers  $U$ .

*Proof.* Disjointness is straightforward. Suppose given  $X \neq U$ . Since the  $(F_\lambda^-, F_\lambda^+)$  cover  $U$ , there is  $\lambda$  with

$$X + F_\lambda^- \neq X + F_\lambda^+.$$

Now letting  $Y = X + F_\lambda^-$  and  $Z = F_\lambda^+$ , we have  $Z \not\subseteq Y$ . Thus since the  $(G_\sigma^-, G_\sigma^+)$  strongly cover  $U$ , there is  $\sigma$  with

$$Y + (G_\sigma^- \cap Z) \neq Y + (G_\sigma^+ \cap Z).$$

Hence

$$X + F_\lambda^- + (G_\sigma^- \cap F_\lambda^+) \neq X + F_\lambda^- + (G_\sigma^+ \cap F_\lambda^+).$$

□

## 7. COMPLETION OF THE PROOF

Let  $V$  be a persistence module with the descending chain condition on images and kernels.

**Lemma 7.1.** *For  $t \in R$ , each of the sets*

- (a)  $\{(\text{Im}_{ct}^-, \text{Im}_{ct}^+) : c \text{ a cut with } t \in c^+\}$ , and
- (b)  $\{(\text{Ker}_{ct}^-, \text{Ker}_{ct}^+) : c \text{ a cut with } t \in c^-\}$

*is disjoint and strongly covers  $V_t$ .*

*Proof.* We prove (a); part (b) is similar. If  $c$  and  $d$  are distinct cuts with  $c^+$  and  $d^+$  containing  $t$ , then exchanging  $c$  and  $d$  if necessary, we may assume that  $c^+ \cap d^- \neq \emptyset$ . Now if  $s \in c^+ \cap d^-$ , then  $s < t$  and

$$\text{Im}_{ct}^+ \subseteq \text{Im } \rho_{ts} \subseteq \text{Im}_{dt}^-,$$

giving disjointness. Now suppose that  $Y, Z \subseteq U$  with  $Z \not\subseteq Y$ . Defining

$$c^- = \{s \in R : \text{Im } \rho_{ts} \cap Z \subseteq Y\}, \quad c^+ = \{s \in R : \text{Im } \rho_{ts} \cap Z \not\subseteq Y\},$$

clearly  $c$  is a cut,  $t \in c^+$ , and

$$Y + (\text{Im}_{ct}^- \cap Z) = Y + \left( \bigcup_{s \in c^-} \text{Im } \rho_{ts} \cap Z \right) = \bigcup_{s \in c^-} (Y + (\text{Im } \rho_{ts} \cap Z)) = Y.$$

By Lemma 2.1, we have  $\text{Im}_{ct}^+ = \text{Im } \rho_{ts}$  for some  $s \in c^+$  with  $s \leq t$ , so

$$Y + (\text{Im}_{ct}^+ \cap Z) = Y + (\text{Im } \rho_{ts} \cap Z) \neq Y,$$

giving the strong covering property. □

*Proof of Theorem 1.2.* For  $I$  an interval and  $t \in I$ , we consider the section  $(F_{It}^-, F_{It}^+)$  of  $V_t$  given by

$$F_{It}^\pm = \text{Im}_{\ell t}^\pm + \text{Ker}_{ut}^\pm \cap \text{Im}_{\ell t}^\pm,$$

where  $\ell$  and  $u$  are the cuts determined by  $I$ . As  $I$  runs through all intervals containing  $t$ , the cuts  $\ell$  and  $u$  run through all cuts with  $t \in \ell^+$  and  $t \in u^-$ . Thus by Lemmas 6.2 and 7.1, the set of sections  $(F_{It}^-, F_{It}^+)$  is disjoint and covers  $V_t$ .

Now by Lemma 5.2 we have  $V_{It}^+ = W_{It} \oplus V_{It}^-$  for all  $t \in I$ . It follows that  $F_{It}^+ = W_{It} \oplus F_{It}^-$ . Thus by Lemma 6.1, the space  $V_t$  is the direct

sum of the spaces  $W_{It}$  as  $I$  runs through all intervals containing  $t$ . Thus  $V$  is the direct sum of the submodules  $W_I$ , and each of these is a direct sum of copies of  $k_I$  by Lemma 5.3.  $\square$

## REFERENCES

- [1] F. Chazal, D. Cohen-Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot, Proximity of persistence modules and their diagrams, in Proceedings of the 25th Annual Symposium on Computational Geometry, Association for Computing Machinery, 2009, pp. 237–246.
- [2] F. Chazal, V. de Silva, M. Glisse and S. Oudot, The structure and stability of persistence modules, preprint arXiv:1207.3674.
- [3] F. Chazal, V. de Silva, S. Oudot, Persistence stability for geometric complexes, preprint arXiv:1207.3885.
- [4] W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (1994), 1641–1674.
- [5] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. 11 (1961), 5–167.
- [6] M. Lesnick, The optimality of the interleaving distance on multidimensional persistence modules, preprint arXiv:1106.5305.
- [7] C. M. Ringel, The indecomposable representations of the dihedral 2-groups, Math. Ann. 214 (1975), 19–34.
- [8] A. Zomorodian and G. Carlsson, Computing persistent homology, Discrete Comput. Geom. 33 (2005), 249–274.

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UK

*E-mail address:* `w.crawley-boevey@leeds.ac.uk`